Measure Theory with Ergodic Horizons
Lecture 2

Generation of algebras and σ-algebras.

Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$ be a collection. Is there a smallest (σ-) algebra containing $\mathcal{C}$? There is at least one (σ-) algebra containing $\mathcal{C}$, namely, $\mathcal{P}(X)$.

Observation. Arbitrary intersections of (σ-) algebras is still a (σ-) algebra, i.e. if $\{\mathcal{A}_i : i \in I\}$ is a collection of (σ-) algebras on X, then $\bigcap_{i \in I} \mathcal{A}_i$ is a (σ-) algebra.

Thus, given $\mathcal{C} \subseteq \mathcal{P}(X)$, $\langle \mathcal{C} \rangle :=$ intersection of all algebras containing $\mathcal{C}$
$\langle \mathcal{C} \rangle_\sigma :=$ intersection of all σ-algebras containing $\mathcal{C}$
is the smallest (σ-) algebra containing.
These are top-down definitions, which are hard to work with, so we seek bottom-up definitions.

Prop. Let $\mathcal{C} \subseteq \mathcal{P}(X)$. Then

(a) $\langle \mathcal{C} \rangle = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, where $\mathcal{C}_0 := \mathcal{C} \cup \{\emptyset\}$, $\mathcal{C}_{n+1} := \{C^c : C \in \mathcal{C}_n\} \cup \{$finite unions of sets in $\mathcal{C}_n\}$.

(b) $\langle \mathcal{C} \rangle_\sigma = \bigcup_{\alpha \in \omega_1} \mathcal{C}_\alpha$, where $\mathcal{C}_0 := \mathcal{C} \cup \{\emptyset\}$, $\mathcal{C}_\alpha := \bigcup_{\beta < \alpha} \{C^c : C \in \mathcal{C}_\beta\} \cup \{$ctbl unions of sets in $\bigcup_{\beta<\alpha} \mathcal{C}_\beta\}$.
ω_1 ← first unctbl ordinal

Proof. (a) is left as HW and (b) is an optional exercise. $\square$

Def. For a metric (more generally topological) space X, the σ-algebra generated by all open sets is called the Borel σ-algebra, and the sets in it are called Borel sets. We denote the Borel σ-algebra of X by $\mathcal{B}(X)$.
Observe that the Borel σ-algebra is also generated by closed sets.

Def. A basis for a metric (topological space) X is a collection $\mathcal{C}$ of open sets such that each open set in X is a union of some sets from $\mathcal{C}$. X is called second ctbl if it admits a ctbl basis.

Examples. (a) In $\mathbb{R}^d$, rational boxes form a basis, so $\mathbb{R}^d$ is 2^{nd} ctbl.
(b) For a ctbl $A \neq \emptyset$, the space $A^{\mathbb{N}}$ is 2^{nd} ctbl because the cylinders form a ctbl basis for $A^{\mathbb{N}}$.

Observation. If $\mathcal{C}$ is a ctbl basis for a metric space X, then $\langle \mathcal{C} \rangle_\sigma = \mathcal{B}(X)$.

Prop. For metric spaces, 2^{nd} ctblity is equivalent to separability.
Proof. HW

Def. A measurable space is a pair $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ is a σ-algebra on X.

Measures.

Def. For a set X and an algebra $\mathcal{A}$ on X, a function $\mu : \mathcal{A} \to [0, \infty]$ is said to be:

- finitely additive if $\mu\left(\bigsqcup_{i \le n} A_i\right) = \sum_{i \le n} \mu(A_i)$ for all disjoint $A_0, \dots, A_n \in \mathcal{A}$.
- ctbly additive if $\mu\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i)$ for all disjoint $A_0, A_1, \dots \in \mathcal{A}$ with $\bigsqcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$.

Def. For a measurable space $(X, \mathcal{S})$, a measure on X is a ctbly additive function $\mu : \mathcal{S} \to [0, \infty]$ such that $\mu(\emptyset) = 0$. A measurable space $(X, \mathcal{S})$ equipped with a measure μ is called a measure space and denoted $(X, \mathcal{S}, \mu)$.

Caution. There is a term *finitely additive measure* which means a function $\mu : \mathcal{A} \to [0, \infty]$ on an algebra $\mathcal{A}$ that is finitely additive and $\mu(\emptyset) = 0$. But finitely additive measures are typically not measures even if $\mathcal{A}$ is a σ-algeb.

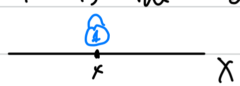
Def. A measure μ on a measurable space $(X, \mathcal{S})$ is called:
- *finite* if $\mu(X) < \infty$.
- *probability* if $\mu(X) = 1$.
- *σ-finite* if X can be partitioned into ctbly many sets from $\mathcal{S}$ each of which having finite measure.

Observation. (a) A ctbl weighted sum of measures is also a measure, i.e. if the μ_n are measures on a measurable space $(X, \mathcal{S})$ and the c_n are non-negative reals then $\sum_{n \in \mathbb{N}} c_n \mu_n$ is also a measure on $(X, \mathcal{S})$.

(b) A ctbl convex combition of probability measures is also a probability measure, i.e. if the μ_n are probability measures on a measurable space $(X, \mathcal{S})$ and the c_n are non-negative reals with $\sum_{n \in \mathbb{N}} c_n = 1$, then $\sum_{n \in \mathbb{N}} c_n \mu_n$ is also a probability measure on $(X, \mathcal{S})$.

Examples. (a) The *zero measure* $\mu_0 \equiv 0$ on any measurable space $(X, \mathcal{S})$.

(b) The *Dirac measure* at a point $x \in X$ is the measure δ_x on $(X, \mathcal{P}(X))$ defined by $\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$.



(c) The *counting measure* on X is the measure μ_c on $(X, \mathcal{P}(X))$ defined by

$$\mu_c(A) := \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise.} \end{cases}$$

Note that if X is ctbl, then $\mu_c = \sum_{x \in X} \delta_x$.

(d) Given a set X, define a measure μ on the σ-algebra of ctbl/co-ctbl sets:

$$\mu(A) := \begin{cases} 1 & \text{if } A \text{ is unctbl} \\ 0 & \text{if } A \text{ is ctbl} \end{cases}.$$

This is a measure due to the fact that ctbl unions of ctbl sets are ctbl. Checking this is left as an exercise.

Def. Let $(X, \mathcal{S}, \mu)$ be a measure space. A set $B \in \mathcal{S}$ is called an atom (or μ-atom) if $\mu(B) > 0$ and $\mu(A) = 0$ or $\mu(A) = \mu(B)$ for every subset $A \subseteq B$ with $A \in \mathcal{S}$.

A measure space $(X, \mathcal{S}, \mu)$ is called

- atomic (or purely atomic) if every set in $\mathcal{S}$ of positive measure contains an atom.
- atomless if it does not have atoms.

Caution. The zero measure μ_0 on any measurable space $(X, \mathcal{S})$ is both atomic and atomless.

All our examples of measures (a)–(d) are atomic. Building non-trivial atomless measures requires work, which we will start next time.